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Zero-dimensional field theory

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Abstract. A study of zero-dimensional theories, based on exact results, is presented. First, relying on a simple diagrammatic representation of the theory, equations involving the generating function of all connected Green's functions are constructed. Second, exact solutions of these equations are obtained for several theories. Finally, renormalization is carried out. Based on the anticipated knowledge of the exact solutions the full dependence on the renormalized coupling constant is studied.

1 Introduction

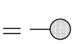
In this paper we study several aspects of zero-dimensional quantum field theory. Such theories may serve as a model (the static ultra-local limit) of more realistic quantum field theories, and as a useful didactic object in their own right, since zero-dimensional theories, for which the path integral is actually a simple integral, allow for many explicit and exact solutions that cannot be obtained in higher dimensions. As recent examples, we may quote 't Hooft [1] and Bender et al. [2]. Questions of particular interest here are the behavior of theories in high orders of perturbation theory (either many loops, or large number of external legs), and of the relation between the diagrammatic perturbation expansion and the full solution. The layout of this paper is as follows. We start by a diagrammatic (re)derivation of equations that govern the set of all connected Green's functions of the theory. We show how for a general scalar theory with arbitrary interactions the Green's functions may be obtained order by order. We point out how the Schwinger-Dyson equation, although derivable from purely diagrammatic arguments, in fact describes a much larger class of solutions. Next, we discuss the representation of these solutions as path integrals over contours in the complex φ-plane. Exact solutions for theories with interactions up to φ^4 are obtained including explicitly perturbative and non-perturbative contributions. We show how a classification of the allowed contours in the complex φ-plane can immediately determine properties of the non-perturbative character of these theories. Renormalization, which for these theories is equivalent to imposing restrictions to diagrams, is also studied. The wave function renormalizations are fully determined and their dependence on the renormalized coupling constant of the theory is presented and discussed.

2 Basic equations

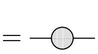
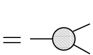
In this section we derive equations for an arbitrary zero-dimensional field theory. The derivation is based entirely upon the diagrammatic representation of the theory. A theory is diagrammatically defined by a sequence of vertices that are weighted by the 'coupling' constants taken for convenience as $-\lambda_k$, for the k-th vertex. In fact, the two-point coupling $\lambda_2 = m^2 \equiv \mu$ can be eliminated by the introduction of the propagator which means that every line of a diagram accounts for a factor $1/\mu$. Moreover a loop in a graph is counted by an additional parameter, $\hbar$. A solution of a zero-dimensional theory is determined by a sequence of objects C_n, $n = 0, 1, 2, \ldots$ that represent the connected, n-point Green's functions, i.e. the sum of all connected diagrams with n external lines. One can define the generating function of the Green's functions as

$$\phi(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} C_{n+1} \quad . \tag{1}$$

We want to write down an equation for ϕ, and in order to do so, we represent it with a diagram:

$\phi(x) =$  .

Derivatives of ϕ with respect to x are represented by extra lines,

$\phi'(x) =$  , $\phi''(x) =$  ,

and so on.

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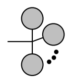
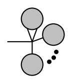
2.1 The Schwinger-Dyson equation

Let us consider a theory with a k-th vertex. In order to write an equation for ϕ we start with the bare vertex and we attach $k-1$ blobs



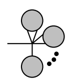
$$\hbar^0 \frac{\phi^{k-1}}{(k-1)!} \; .$$

The factor $1/(k-1)!$ is due to the $k-1$ identical blobs. Considering a one loop attachment we similarly have

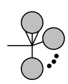


$$\hbar^1 \frac{\phi'}{2!} \frac{\phi^{k-3}}{(k-3)!} \; .$$

The factor $1/(k-2)!$ is again due to the $k-2$ identical blobs, whereas the $1/2!$ is due to the symmetry factor of blob with two lines. Following this reasoning we can proceed with higher powers of $\hbar$. For instance at the two-loop level we get two terms

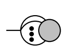


$$\hbar^2 \frac{\phi'^2}{2!2!2!} \frac{\phi^{k-5}}{(k-5)!}$$



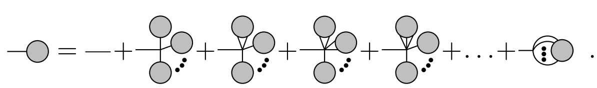
$$\hbar^2 \frac{\phi''}{3!} \frac{\phi^{k-4}}{(k-4)!} \; .$$

Finally the last term, i.e the term with the largest number of loop attachments, will simply read



$$\hbar^{k-2} \frac{\phi^{(k-2)}}{(k-1)!} \; .$$

The result looks like

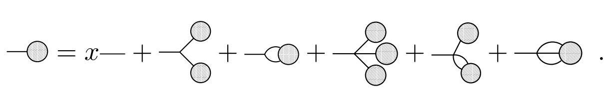


The equation reads

$$x = \mu\phi + \lambda_k \left(\frac{\phi^{k-1}}{(k-1)!} + \hbar^1 \frac{\phi'}{2!} \frac{\phi^{k-3}}{(k-3)!} + \hbar^2 \frac{\phi'^2}{2!2!2!} \frac{\phi^{k-5}}{(k-5)!} + \hbar^2 \frac{\phi''}{3!} \frac{\phi^{k-4}}{(k-4)!} + \ldots + \hbar^{k-2} \frac{\phi^{(k-2)}}{(k-1)!} \right) . \quad (2)$$

For an arbitrary theory a sum over k should be understood. It represents a non-linear differential equation for ϕ, the Schwinger-Dyson (SD) equation, which has been derived by the direct application of the Feynman rules.

In order to be more specific let us consider a theory with only a 3-point and a 4-point vertex. Following the abovementioned reasoning a diagrammatic equation for ϕ looks like



This reads

$$\phi(x) = \frac{x}{\mu} - \frac{\lambda_3}{2\mu} \left[\phi(x)^2 + \hbar\phi'(x) \right] - \frac{\lambda_4}{6\mu} \left[\phi(x)^3 + 3\hbar\phi(x)\phi'(x) + \hbar^2\phi''(x) \right] \; . \quad (3)$$

This equation generates equations for the Green's functions if the power series of $\phi(x)$ is inserted. The first three are

$$\begin{aligned}
C_1 &= -\frac{1}{6\mu} C_1^2(\lambda_4 C_1 + 3\lambda_3) - \frac{\hbar}{2\mu} C_2(\lambda_4 C_1 + \lambda_3) \\
&\quad - \frac{\hbar^2}{6\mu} \lambda_4 C_3 \; , \\
C_2 &= -\frac{1}{2\mu}(2\lambda_3 C_1 C_2 - 2 + \lambda_4 C_1^2 C_2) \\
&\quad - \frac{\hbar}{2\mu}(\lambda_3 C_3 + \lambda_4 C_1 C_3 + \lambda_4 C_2^2) - \frac{\hbar^2}{6\mu} \lambda_4 C_4 \; , \\
C_3 &= -\frac{1}{2\mu}(2\lambda_3 C_1 C_3 + 2\lambda_3 C_2^2 + \lambda_4 C_3 C_1^2 + 2\lambda_4 C_2^2 C_1) \\
&\quad - \frac{\hbar}{2\mu}(3\lambda_4 C_2 C_3 + \lambda_3 C_4 + \lambda_4 C_1 C_4) - \frac{\hbar^2}{6\mu} \lambda_4 C_5 \; . \quad (4)
\end{aligned}$$

The SD equation is invariant under certain redefinition of the parameters involved. It is not difficult to prove that if $\phi(\mu, \lambda_k, \hbar; x)$ is a solution, also $c^\beta \phi(c^{\alpha-2\beta}\mu, c^{\alpha-k\beta}\lambda_k, c^\alpha\hbar; c^{\alpha-\beta}x)$ is a solution for any c, α, β. This scaling property is also a concequence of the fact that $\phi(\mu, \lambda_k, \hbar; x)/\sqrt{\hbar/\mu}$ is a dimensionless function of the scaled variables $y = x/\sqrt{\hbar\mu}$ and $g_k = \lambda_k \hbar^{k/2-1}/\mu^{k/2}$. The scaling property can be expressed with the following equations, derived, for instance, by differentiating with respect to c and taking $c = 1$:

$$\begin{aligned}
&\left(x\frac{\partial}{\partial x} + \mu\frac{\partial}{\partial\mu} + \hbar\frac{\partial}{\partial\hbar} + \lambda_k\frac{\partial}{\partial\lambda_k} \right)\phi = 0 \; , \\
&\left(1 + x\frac{\partial}{\partial x} + 2\mu\frac{\partial}{\partial\mu} + k\lambda_k\frac{\partial}{\partial\lambda_k} \right)\phi = 0 \; . \quad (5)
\end{aligned}$$

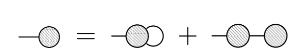
These equations are equivalent to the usual topological relations that relate the number of external lines E, the number of internal lines I, the number of k-vertices V_k, and the number of loops L, appearing in any diagram,

$$kV_k = E + 2I \qquad V_k = I + 1 - L \; .$$

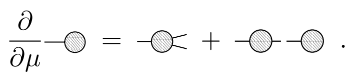
A sum over k should be understood in the general case.

2.2 Stepping equations

In the diagrammatic construction, one assumes that every Green's function can be written as a sum of diagrams, consisting of vertices connected by lines (propagators). The power of $1/\mu$ in a diagram is equal to the number of propagators, and hence the operation $-\partial/\partial\mu$ on this diagram corresponds to cutting a single propagator in all possible places in that diagram. There are two possibilities for the result: the chosen propagator may be either a part of a loop, in which case the diagram remains connected when we cut this line, or part of the 'tree skeleton', such that cutting it makes the diagram diconnected:



In the first case the cut diagram remains connected but gains two external lines at the price of one loop (*i.e.* on power of $\hbar$); in the second place, the cut diagram falls apart into two connected diagrams:



Putting in the correct symmetry factors, we can express this procedure by

$$\text{S2:} \qquad \frac{\partial}{\partial\mu}\phi(x) + \phi(x)\phi'(x) + \frac{\hbar}{2}\phi''(x) = 0 \ , \tag{6}$$

where the second term comes from diagrams that fall apart under cutting, and the third one from loops that are cut open. We call (6) the Step-2 equation (S2) since it describes a procedure in which the number of external legs is increased in steps of 2.

Like SD (3), the Stepping equation S2 implies relations between various C's. The lowest few of these read

$$\begin{aligned}
C_3 &= -\frac{2}{\hbar}\left(C_1C_2 + \frac{\partial}{\partial\mu}C_1\right) \ , \\
C_4 &= -\frac{2}{\hbar}\left(C_1C_3 + C_2^2 + \frac{\partial}{\partial\mu}C_2\right) \ , \\
C_5 &= -\frac{2}{\hbar}\left(C_1C_4 + 3C_2C_3 + \frac{\partial}{\partial\mu}C_3\right) \ , \\
C_6 &= -\frac{2}{\hbar}\left(C_1C_5 + 4C_2C_4 + C_3^2 + \frac{\partial}{\partial\mu}C_4\right) \ ,
\end{aligned} \tag{7}$$

and so on. Note that S2 is completely independent of the interaction potential, and therefore perforce contains information independent of that contained in the SD. It follows that there must be solutions to SD that do *not* obey S2 and that these solutions cannot be represented by Feynman diagrams.

It is possible to combine SD and S2 in the following manner. Taking the first equation in (3), we express C_3 in C_2 and C_1, and solve for C_2:

$$C_2 = \frac{2\lambda_4\dfrac{\partial C_1}{\partial\mu} - \dfrac{1}{\hbar}\left(6\mu C_1 + 3\lambda_3C_1^2 + \lambda_4C_1^3\right)}{3\lambda_3 + \lambda_4C_1} \ .$$

Inserting this into the second equation of (3), we find a differential equation for C_1 alone:

$$\begin{aligned}
0 &= 4\hbar^2\lambda_4^3\left(\frac{\partial C_1}{\partial\mu}\right)^2 - 2\hbar^2\lambda_4^2\frac{\partial^2C_1}{\partial\mu^2}\left(\lambda_4C_1 + 3\lambda_3\right) - \hbar\frac{\partial C_1}{\partial\mu} \\
&\times\left[9\lambda_3^2(3\lambda_3 + \lambda_4C_1) + 6\mu\lambda_4^2C_1 - 36\mu\lambda_3\lambda_4\right] - 9\mu\lambda_3C_1^2 \\
&\times(3\lambda_3 + \lambda_4C_1) + 3\hbar\lambda_4^2C_1^2 - 54\mu^2\lambda_3C_1 - 27\hbar\lambda_3^2 \ .
\end{aligned} \tag{8}$$

By inserting the series expansion

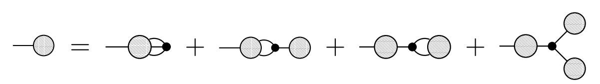
$$C_1 = \sum_{k\geq1}\alpha_k\hbar^k \ ,$$

we can then successively determine the coefficients:

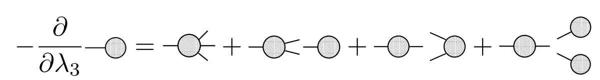
$$\begin{aligned}
\alpha_1 &= -\frac{1}{2\mu^2}\lambda_3 \ , \\
\alpha_2 &= -\frac{1}{24\mu^5}(15\lambda_3^3 - 16\mu\lambda_3\lambda_4) \ , \\
\alpha_3 &= -\frac{1}{48\mu^8}(90\lambda_3^5 - 185\mu\lambda_3^3\lambda_4 + 66\mu^2\lambda_3\lambda_4^2) \ , \\
\alpha_4 &= -\frac{1}{1152\mu^{11}}(9945\lambda_3^7 - 30270\mu\lambda_3^5\lambda_4 \\
&\quad + 24280\mu^2\lambda_3^3\lambda_4^2 - 4352\mu^3\lambda_3\lambda_4^3) \ ,
\end{aligned}$$

and so on.

Whereas S2 is independent of the interaction potential, we can also derive stepping equations by deleting vertices rather than cutting lines. For example, let us depict all possible ways in which a selected φ^3 vertex (denoted by a dot) can occur in a connected graph:



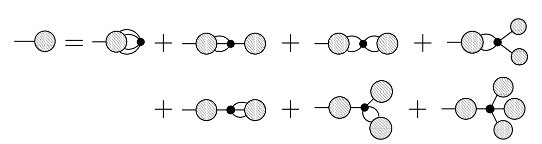
Deleting this vertex gives us



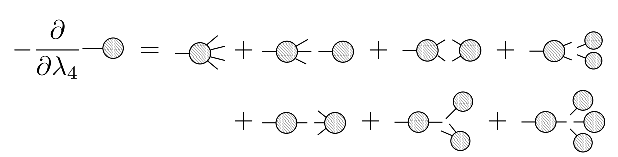
or, in terms of $\phi(x)$, the following Step-3 equation (S3):

$$\text{S3:} \qquad \frac{\partial}{\partial\lambda_3}\phi + \frac{1}{6}\hbar^2\phi''' + \frac{1}{2}\hbar\left[\phi\phi'' + (\phi')^2\right] + \frac{1}{2}\phi^2\phi' = 0 \ . \tag{9}$$

A similar treatment holds for φ^4 vertices: the possible ways in which such a vertex can occur is given by



and the result of deleting is given by



The corresponding Step-4 equation (S4) is

$$\begin{aligned}
\text{S4:} \qquad &\frac{\partial}{\partial\lambda_4}\phi + \frac{1}{24}\hbar^3\phi'''' + \frac{1}{12}\hbar^2\left[2\phi\phi''' + 5\phi'\phi''\right] \\
&+\frac{1}{4}\hbar\left[\phi^2\phi'' + 2\phi(\phi')^2\right] + \frac{1}{6}\phi^3\phi' = 0 \ .
\end{aligned} \tag{10}$$

2.3 The charged scalar field

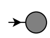
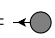
Up to now we dealt with diagrammatic construction of zero-dimensional field theories involving only one field. As an illustrative extension, we consider a theory with two fields, *i.e.* a complex, or charged, scalar field. The Green's functions are labeled with two integers, and the generating function has two expansion parameters x and $\bar{x}$. Let us introduce the notation

$$\partial := \frac{\partial}{\partial x} \qquad , \qquad \bar{\partial} := \frac{\partial}{\partial \bar{x}} \ ,$$

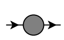
then

$$\phi(x,\bar{x}) = \sum_{n,m=0}^{\infty} \frac{x^n}{n!}\frac{\bar{x}^m}{m!} C_{n,m+1} \ .$$

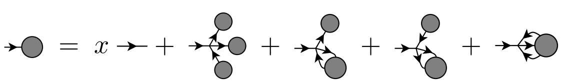
To write down the SD equation, we introduce two kind of lines, distinguishable by an arrow. The generating function is represented by

$\phi(x,\bar{x}) =$  , $\bar{\phi}(x,\bar{x}) =$  .

An incoming external line represents a $\bar{\partial}$, and an outgoing line represents a ∂. Notice that

$\hbar\bar{\partial}\phi = \hbar\partial\bar{\phi} =$  .

We also introduce a four point vertex with two incoming and two outgoing lines, so that the SD equation we want ϕ to satisfy is given by

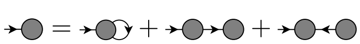
 ,

or

$$\phi = \frac{x}{\mu} - \frac{\lambda}{2\mu}\phi^2\bar{\phi} - \frac{\lambda\hbar}{\mu}\phi\partial\phi - \frac{\lambda\hbar}{2\mu}\bar{\phi}\bar{\partial}\phi - \frac{\lambda\hbar^2}{2\mu}\partial\bar{\partial}\phi \ . \quad (11)$$

Notice that incoming and outgoing lines are not equivalent, which is represented by the symmetry factors.

Also for the charged scalar field, we can write down stepping equations. For the first one, we use that in the diagrammatic interpretation

 ,

leading to

$$\frac{\partial}{\partial\mu}\phi = -\hbar\partial\bar{\partial}\phi - \phi\partial\phi - \bar{\phi}\bar{\partial}\phi \ . \quad (12)$$

The diagrammatic derivation of the stepping equation involving the derivative with respect to λ, although equally straightforward, is rather cumbersome, leading to many terms which we refrain from listing here.

3 Solutions to the equations

3.1 The integral representation

The SD equation is highly non-linear. Let us consider the general term connected with the coupling λ_k in (2):

$$Q_k = \frac{\phi^{k-1}}{(k-1)!} + \hbar^1\frac{\phi'}{2!}\frac{\phi^{k-3}}{(k-3)!} + \hbar^2\frac{\phi'^2}{2!2!2!}\frac{\phi^{k-5}}{(k-5)!} + \hbar^2\frac{\phi''}{3!}\frac{\phi^{k-4}}{(k-4)!} + \ldots + \hbar^{k-2}\frac{\phi^{(k-2)}}{(k-1)!} \ .$$

It can obviously be organized such that it can be written as

$$Q_k = \sum_{m=1}^{k-1}\sum_{\{\mathbf{a}_{k-1};m\}} \frac{\hbar^{k-m-1}}{(1!)^{a_1}a_1!(2!)^{a_2}a_2!\cdots((k-1)!)^{a_{k-1}}a_{k-1}!} \times(\phi)^{a_1}(\phi')^{a_2}\ldots(\phi^{(k-2)})^{a_{k-1}} \ , \quad (13)$$

where $\sum_{\{\mathbf{a}_{k-1};m\}}$ stands for the summation with $a_1, a_2, \ldots, a_{k-1}$ running over all positive integers under the restrictions that

$$a_1 + 2a_2 + 3a_3 + \ldots + (k-1)a_{k-1} = k-1$$

and

$$a_1 + a_2 + \ldots + a_{k-1} = m \ .$$

This sum can be interpreted following the time-honored formula of *Faà di Bruno* [3]:

$$\frac{d^n}{dx^n} f(\,g(x)\,) = \sum_{m=0}^{n} f^{(m)}(\,g(x)\,)\sum_{\{\mathbf{a}_n;m\}}(n;a_1,\ldots,a_n) \times\{g'(x)\}^{a_1}\{g''(x)\}^{a_2}\cdots\{g^{(n)}(x)\}^{a_n}$$

where

$$(n;a_1,\ldots,a_n) = \frac{n!}{(1!)^{a_1}a_1!(2!)^{a_2}a_2!\cdots(n!)^{a_n}a_n!} \ .$$

The identifications $g'(x) = \phi(x)$ and $f^{(m)}(\,g(x)\,) = \hbar^{-m} f(\,g(x)\,)$ with the solution

$$g(x) = \int dx\,\phi(x) \quad ,$$

$$f(\,g(x)\,) = R(x) = \exp\left(\frac{1}{\hbar}\int dx\,\phi(x)\right) \quad , \quad (14)$$

lead to an equation for R, which, including all possible vertices, reads

$$\sum_{k=3}^{\infty}\frac{\lambda_k}{(k-1)!}\hbar^{k-1}R^{(k-1)} + \mu\hbar R'(x) - (x-\lambda_1)R(x) = 0 \ . \quad (15)$$

This is a linear equation, and a solution can be represented by an integral

$$R_\Gamma(x) = \int_\Gamma d\varphi\,\exp\left\{\frac{1}{\hbar}[x\varphi - S(\varphi)]\right\} \ , \quad (16)$$

where

$$S(\varphi) = \lambda_1\varphi + \frac{1}{2}\mu\varphi^2 + \sum_{k=3}^{\infty}\frac{\lambda_k}{k!}\varphi^k \ ,$$

and where Γ is a contour in the complex φ-plane, such that the difference between the values of the integrand in the end-points is zero. This is the well known *path integral representation*, with the *action* S.

A remark is in order. Although the SD equations resulted from a a purely diagrammatic construction, their solutions, expressed through the path integral representation, include non-perturbative ones that cannot be realized in a weak coupling expansion, as we will see below.

Secondly, we note that (15) can be used to write the original SD equation for ϕ compactly as

$$x = \lambda_1 + \mu\phi + \sum_{k\geq 3} \frac{\lambda_k}{(k-1)!} \left(\hbar \frac{\partial}{\partial x} + \phi \right)^{k-2} \phi \ . \tag{17}$$

For a general interacting theory, differentiating R_Γ with respect to λ_k in (16), the stepping equation in terms of ϕ can be rewritten as

$$\frac{\partial \phi}{\partial \lambda_k} = -\frac{1}{k!} \frac{\partial}{\partial x} \left(\phi + \hbar \frac{\partial}{\partial x} \right)^{k-1} \phi \tag{18}$$

and in case only a k-vertex and the tadpole λ_1 is present, combining with the SD a simpler form is obtained

$$\frac{\partial \phi}{\partial \lambda_k} = -\frac{1}{k\lambda_k} \left((x - \lambda_1)\, \phi' + \phi - 2\mu\phi\phi' - \hbar\mu\phi'' \right) . \tag{19}$$

Moreover for a charged scalar field the stepping equation in terms of ϕ can be written in a compact form

$$\frac{\partial \phi}{\partial \lambda} = -\frac{1}{4} \bar{\partial} (\bar{\phi} + \hbar\partial)^2 (\phi + \hbar\bar{\partial}) \phi \, . \tag{20}$$

Finally, the linear SD equation for $\varphi^3 + \varphi^4$-theory becomes simply

$$\frac{1}{6}\lambda_4 \hbar^3 R'''(x) + \frac{1}{2}\lambda_3 \hbar^2 R''(x) + \mu\hbar R'(x) - xR(x) = 0 \ , \tag{21}$$

and we see that $R(x)$ admits 3 linearly independent solutions (2 if $\lambda_4 = 0$). Hence $\phi(x)$ has a 2-parameter family of solutions (a 1-parameter family if $\lambda_4 = 0$). In the sequel we will show how to get exact explicit solutions for a number of scalar theories.

3.2 Results for pure φ^3-theory

In this section we derive results for the pure φ^3-theory, with action

$$S(\varphi) = \frac{1}{2}\mu\varphi^2 + \frac{1}{6}\lambda\varphi^3 \ . \tag{22}$$

This theory is interesting because as we will see the solution for the generating function can be expressed directly in terms of known special functions. Defining

$$y = \frac{x}{\sqrt{\hbar\mu}} \quad , \qquad \xi = \frac{\lambda\sqrt{\hbar}}{6\mu^{3/2}}$$

the SD equation becomes

$$3\xi R''(y) + R'(y) - yR(y) = 0 \tag{23}$$

which admits the following general solution

$$R(y) = e^{-y/6\xi} \left[c_1 \mathrm{Ai}(t) + c_2 \mathrm{Bi}(t) \right] \tag{24}$$

where

$$t = (3\xi)^{-1/3} \left(\frac{1}{12\xi} + y \right) .$$

Ai and Bi are the Airy functions (cf. [3]). The solution for the generating function of connected Green's functions is given by

$$\phi(x) = \sqrt{\frac{\hbar}{\mu}} \left(-\frac{1}{6\xi} + 2^{1/2} t_0^{1/4} \frac{\mathrm{Ai}'(t) + K\mathrm{Bi}'(t)}{\mathrm{Ai}(t) + K\mathrm{Bi}(t)} \right) \tag{25}$$

with $t_0 = t(y = 0)$. The constant K is not determined by the SD equation: in fact it could have been even a function of ξ.

For solutions that admit a diagrammatic representation extra information can be obtained by combining SD and stepping equations. For instance, the scaling and stepping equations of the previous section result to a K that is independent of ξ. Moreover by combining SD and S2 an equation involving only C_1:

$$2\mu^2 C_1 + \lambda\mu C_1^2 + \hbar\lambda^2 \frac{\partial}{\partial\mu} C_1 + \hbar\lambda = 0 \ , \tag{26}$$

can be obtained. The series of substitutions

$$v = \frac{\hbar\lambda^2}{\mu^3} \ , \ \ C_1 = -\frac{\lambda\hbar}{2\mu^2} f(v) \ , \ \ w = \frac{1}{3v} \ ,$$
$$f(v) = -\frac{2k'(w)}{vk(w)} \ , \ \ k(w) = w^{1/3} e^{-w} \psi(w) \ ,$$

leads to the Bessel equation

$$w^2 \psi''(w) + w\psi'(w) - \left(w^2 + \frac{1}{9} \right) \psi(w) = 0 \ .$$

The special solution choice $\psi(w) = K_{1/3}(w)$ gives the following tadpole and its asymptotic expansion:

$$f(v) = \frac{2}{v} \left(\frac{K_{2/3}(w)}{K_{1/3}(w)} + 1 \right) \ ,$$
$$C_1 \sim -2\frac{\mu}{\lambda} - \frac{\hbar\lambda^2}{2\mu^2} \left(1 - \frac{5}{4}v + \frac{15}{4}v^2 - \frac{1105}{64}v^3 + \frac{1695}{16}v^4 + \cdots \right) \ . \tag{27}$$

This tadpole, therefore, has a non-perturbative contribution. The more generic choice $\psi = k_1 I_{1/3}(w) + k_2 I_{-1/3}(w)$, with $k_1 \neq -k_2$, gives

$$f(v) = -\frac{2}{v} \left(\frac{k_1 I_{-2/3}(w) + k_2 I_{2/3}(w)}{k_1 I_{1/3}(w) + k_2 I_{-1/3}(w)} - 1 \right) \ ,$$

$$C_1 \sim -\frac{\hbar\lambda^2}{2\mu^2}\left(1+\frac{5}{4}v+\frac{15}{4}v^2+\frac{1105}{64}v^3 +\frac{1695}{16}v^4+\cdots\right) \quad , \tag{28}$$

which is the standard perturbative result [4]. The coefficients k_1 and k_2 drop out for the perturbative expansion: they simply account for non-perturbative contributions that are not computable perturbatively!

A remark is in order here: although all the terms in the perturbative series for C_1 have strictly the same complex phase and according to the traditional wisdom the series is not Borel summable, the exact result is well defined, indicating that a suitable generalization of the Borel transform will produce the right answer [5].

Another interesting aspect is the large n behavior of the Green's functions, where n refers to the number of external legs, a problem that is traditionally seen as relevant to the unitarity of the S matrix [6]. This can be traced from the analytical structure of the solution for the generating functions in the complex x-plane. As is evident from the fact that the solution, (24), for the generating function of all connected and disconnected graphs is an entire function, the corresponding Green's function Z_n grows slower than $n!$; in fact it grows like $(n!)^{2/3}$. On the other hand the C_n, the connected graphs, exhibit a factorial growth, since their generating function $\phi(x)$ posesses poles at finite complex values of x.

3.3 Results for pure φ^4-theory

In this section we derive the lowest Green's functions for the pure φ^4 theory, with action

$$S(\varphi)=\frac{1}{2}\mu\varphi^2+\frac{1}{24}\lambda\varphi^4 \quad . \tag{29}$$

Defining

$$y=\frac{x}{\sqrt{\hbar\mu}} \quad , \qquad \xi=\frac{\lambda\sqrt{\hbar}}{24\mu^2}$$

we get for the SD equation

$$4\xi R'''(y)+R'(y)-yR(y)=0 \quad . \tag{30}$$

There are three solutions, which can be represented as follows:

$$\begin{aligned} R_1(y) &= \sum_{n=0}^{\infty}\frac{y^{2n}}{n!}\,(32\xi)^{-n/2}\,\mathrm{U}(n;(8\xi)^{-1/2}) \\ R_2(y) &= \sum_{n=0}^{\infty}(-1)^n\frac{y^{2n}}{n!}\,(32\xi)^{-n/2}\,\frac{\mathrm{V}(n;(8\xi)^{-1/2})}{\Gamma(n+\frac{1}{2})} \\ R_3(y) &= \sum_{n=0}^{\infty}\frac{y^{2n+1}}{(2n+1)!}\,(4\xi)^{-n/2}\,i^n\mathrm{H}_n(i(16\xi)^{-1/2}) \quad , \end{aligned} \tag{31}$$

where $\mathrm{U}(\nu;x)$ and $\mathrm{V}(\nu;x)$ are the parabolic cylinder functions, and H_n is the n^{th} Hermite polynomial (cf. [3]). The general solution is a linear combination with arbitrary coefficients. As we can immediately see contrary to what is argued in many standard textbooks the odd Green's functions do not necessarily vanish.

On the other hand on can study the S2 equation as well. For this we have to distinguish two possible cases: the 'standard' one, with C_1 and the higher odd Green's functions vanishing, and the case where $C_1 \neq 0$.

Let us consider the first case with a zero tadpole. In this case we cannot, of course, directly use the results derived above, since these deal with C_1. The S2 becomes somewhat simpler, and in particular

$$C_4=-\frac{2}{\hbar}\left(C_2^2+\frac{\partial}{\partial\mu}C_2\right) \quad .$$

On dimensional grounds we see that we can write

$$C_2=\frac{1}{\mu}\beta(v) \quad , \qquad v=\frac{\lambda\hbar}{\mu^2} \quad ,$$

where v is dimensionless. Inserting all this into the first nonzero term (that with x^1) in SD, we find the following equation for β:

$$4v^2\beta'(v)+v\beta(v)^2+(2v+6)\beta(v)-6=0 \quad .$$

The substitutions

$$\beta(v)=4v\frac{g'(v)}{g(v)} \quad , \quad g(v)=v^{-1/4}e^w\psi(w) \quad , \quad w=\frac{3}{4v} \ , \tag{32}$$

lead then to

$$w^2\psi''(w)+w\psi'(w)-\left(w^2+\frac{1}{16}\right)\psi(w)=0 \quad , \tag{33}$$

which has the modified Bessel functions for its solutions. The general solution can always be written as

$$\psi(w)=k_1I_{1/4}(w)+k_2I_{-1/4}(w) \quad .$$

It is instructive to consider the perturbative form of these results, that is, the limit where $\hbar$ becomes infinitesimally small, or w goes to infinity. Since $I_{1/4}$ and $I_{-1/4}$ have the same asymptotic expansion, a generic choice of $k_{1,2}$ will lead to a single perturbative expansion. The single exception is the choice $k_1=-k_2$ which leads to $\psi(w)\propto K_{1/4}(w)$, with asymptotic expansion

$$\begin{aligned}\beta(v) &= \frac{3}{v}\left(\frac{K_{3/4}(w)}{K_{1/4}(w)}-1\right) \\ &\sim 1-\frac{1}{2}v+\frac{2}{3}v^2-\frac{11}{8}v^3+\frac{34}{9}v^4+\cdots \quad .\end{aligned}$$

This is the standard perturbative expansion, in which the propagator starts with $\frac{1}{\mu}$, and has loop corrections in powers of $\hbar$:

$$C_2=C_2^{(1)}=\frac{1}{\mu}\left(1-\frac{\lambda\hbar}{2\mu^2}+\frac{2\lambda^2\hbar^2}{3\mu^4}+\cdots\right) \quad . \tag{34}$$

The alternating signs are of course due to the fact that the Feynman rules prescribe a factor $-\lambda$ for each vertex in our Euclidean model. The asymptotic expansion in all other cases is equal to that for the choice $k_2 = 0$, for which we find

$$\beta(v) = -\frac{3}{v}\left(\frac{I_{-3/4}(w)}{I_{1/4}(w)} + 1\right) \sim -\frac{6}{v} + 1 + \frac{1}{2}v + \frac{2}{3}v^2 + \frac{11}{8}v^3 + \frac{34}{9}v^4 + \cdots \quad ,$$

which gives a nonstandard expansion:

$$C_2 = C_2^{(2)} = -\frac{6\mu}{\lambda\hbar} + \frac{1}{\mu}\left(1 + \frac{\lambda\hbar}{2\mu^2} + \frac{2\lambda^2\hbar^2}{3\mu^4} + \cdots\right) \quad . \quad (35)$$

Note the occurrence of a 'non-perturbative' term $1/\lambda$ here: the rest of the expansion has an apparent opposite sign of the coupling constant. An other way to look at this solution is by examining the saddle point equation, $\delta S/\delta\phi = x$: the abovementioned solution corresponds to the saddle point $\phi_c = \sqrt{-6\mu/\lambda} + \mathcal{O}(x)$.

In the case $C_1 \neq 0$ we can write, again on dimensional grounds,

$$C_1 = \alpha(v)\sqrt{\frac{\mu}{\lambda}} \quad , \qquad C_2 = \frac{1}{\mu}\beta(v) \quad ,$$

with v as before. The first term (with x^0) in SD now gives us a relation between α and β:

$$\beta(v) = \frac{1}{v\alpha(v)}\left((6-v)\alpha(v) + 4v^2\alpha'(v)\right) \quad ,$$

and then the second term (x^1) gives

$$16v^2\alpha(v)\alpha''(v) - 32v^2\alpha'(v)^2 + (32v - 24)\alpha(v)\alpha'(v) - 3\alpha(v)^2 = 0 \quad .$$

Using w as before, we may now substitute

$$\alpha(v) = \frac{e^w\sqrt{v}}{\psi(w)} \quad ,$$

to find that $\psi(w)$ again obeys the Bessel equation, (33). For the asymptotic expansions, again two distinct choices are possible. First, the choice

$$\psi(w) = \frac{1}{p}K_{1/4}(w)$$

gives

$$\alpha(v) = \frac{pe^w\sqrt{v}}{K_{1/4}(w)} \quad ,$$
$$\beta(v) = \frac{3}{v}\left(\frac{K_{3/4}(w)}{K_{1/4}(w)} - 1\right) - \frac{p^2e^{2w}}{K_{1/4}(w)^2} \quad ,$$

and the following asymptotic forms for $C_{1,2}$:

$$C_1 \sim p\sqrt{\frac{3\mu}{2\pi\lambda}}e^{2w} \quad , \qquad C_2 \sim C_2^{(1)} + \frac{2p^2we^{2w}}{\mu\pi} \quad . \quad (36)$$

The alternative choice, for which we may take

$$\psi(w) = \frac{1}{p}I_{1/4}(w) \quad ,$$

leads to

$$\alpha(v) = \frac{pe^w\sqrt{v}}{I_{1/4}(w)} \quad ,$$
$$\beta(v) = -\frac{3}{v}\left(\frac{I_{-3/4}(w)}{I_{1/4}(w)} + 1\right) - \frac{p^2e^{2w}}{I_{1/4}(w)^2} \quad ,$$

and

$$C_1 \sim p\sqrt{\frac{3\pi\mu}{2\lambda}} \quad ,$$
$$C_2 \sim C_2^{(2)} - \frac{2\pi p^2 v}{\mu}\left(1 - \frac{1}{4}v - \frac{13}{96}v^2 - \frac{73}{384}v^3 - \cdots\right) . (37)$$

In contrast to the zero-tadpole case, there remains an arbitrary parameter in these solutions, p: it reflects the presence of the 'non-perturbative' tadpole-like contribution and has to be determined by additional requirements.

3.4 Results for $\varphi^3 + \varphi^4$-theory

For the general zero-dimensional φ^3+φ^4-theory, the action is given by

$$S(\varphi) = \frac{1}{2}\mu\varphi^2 + \frac{1}{6}\lambda_3\varphi^3 + \frac{1}{24}\lambda_4\varphi^4 \quad . \quad (38)$$

In the dimensionless variables

$$y = \frac{x}{\sqrt{\mu\hbar}} \quad , \quad g_3 = \frac{\lambda_3}{\mu}\sqrt{\frac{\hbar}{\mu}} \quad , \quad g_4 = \frac{\lambda_4\hbar}{\mu^2} \quad ,$$

the SD equation becomes

$$\frac{1}{6}g_4R'''(y) + \frac{1}{2}g_3R''(y) + R'(y) - yR(y) = 0 \quad . \quad (39)$$

To solve this equation, let

$$R(y) = e^{-yg_3/g_4}F(y) \quad .$$

Then F satisfies the equation

$$\frac{1}{6}g_4F'''(y) + \alpha F'(y) - (y+\beta)F(y) = 0 \quad , \quad (40)$$

where

$$\alpha = 1 - \frac{g_3^2}{2g_4} \quad , \qquad \beta = \frac{g_3}{g_4}\left(1 - \frac{g_3^2}{3g_4}\right) \quad .$$

Finally, changing variables

$$y + \beta = \frac{\eta}{\sqrt{\alpha}} \quad , \qquad 4\xi = \frac{g_4}{6\alpha^2} = \frac{g_4}{6}\left(1 - \frac{g_3^2}{2g_4}\right)^{-2} \quad ,$$

(40) becomes

$$4\xi F'''(\eta) + F'(\eta) - \eta F(\eta) = 0 \quad . \quad (41)$$

(41) is exactly (30) of the pure φ^4-theory, so that the solutions here are those of (31) with ξ as given above and y replaced by η.

3.5 Results for the charged scalar field

For the complex scalar field, the path integral solution is given by

$$R(x,\bar{x}) = \int d\varphi d\bar{\varphi}\, \exp\left\{\frac{1}{\hbar}[x\bar{\varphi} + \bar{x}\varphi - S(\varphi,\bar{\varphi})]\right\} \quad ,$$
$$S(\varphi,\bar{\varphi}) = \mu\bar{\varphi}\varphi + \frac{\lambda}{4}\,(\bar{\varphi}\varphi)^2 \quad . \tag{42}$$

Due to charge conservation (O(2)-symmetry) one can easily show that R only depends on the modulus $x\bar{x}$, so that it satisfies the following equation

$$\zeta R'''(\zeta) + 2R''(\zeta) + \alpha^2[R'(\zeta) - R(\zeta)] = 0 \quad ,$$
$$\zeta = \frac{x\bar{x}}{\mu\hbar} \quad , \quad \alpha = \mu\left(\frac{2}{g\hbar}\right)^{1/2} \quad . \tag{43}$$

The third order equation can be solved by power series expansion in ζ and two of its solutions are given by

$$R_1(\zeta) = \sum_{n=0}^{\infty} \frac{1}{i(n!)^2}\left(\frac{i\zeta\alpha}{\sqrt{2}}\right)^n \mathrm{H}_n\left(\frac{i\alpha}{\sqrt{2}}\right) \quad ,$$
$$R_2(\zeta) = \sum_{n=0}^{\infty} \frac{(\zeta\alpha)^n}{n!}\,\mathrm{U}(n+\tfrac{1}{2},\alpha) \quad , \tag{44}$$

where H_n stands for the n^{th} order Hermite polynomial and $\mathrm{U}(\nu,x)$ is the parabolic cylinder function. The third solution can be found by standard procedures but the actual result is rather cumbersome and we refrain from giving it explicitly.

Where R_1 is a purely non-perturbative solution, R_2 has an asymptotic series expansion which leads to the normal perturbation series for the connected Green's functions: for instance the two point function is given by

$$C_{1,1} = \frac{1}{\mu} - \frac{g\hbar}{\mu^3} + \frac{5}{2}\frac{g^2\hbar^2}{\mu^5} + \mathcal{O}(g^3\hbar^3) \quad . \tag{45}$$

Although the standard integral representation of R is given by (42), the differential equation in the variable ζ leads to another peculiar single contour integral representation

$$R(\zeta) = \int_\Gamma d\psi\, \exp\left(\zeta\psi - \ln\psi - \frac{\alpha^2}{\psi} + \frac{\alpha^2}{2\psi^2}\right) \quad ,$$

where the contour Γ is from infinity to infinity such that the integral is convergent.

3.6 Contours in the integral representation

The integral representation of the solutions for the pure φ^3-theory, with $\psi = \sqrt{\mu/\hbar}\,\varphi$, can be written as

$$R(y;\xi) = K\int_\Gamma d\psi\, \exp\left(-\frac{1}{2}\psi^2 - \xi\psi^3 + y\psi\right) \quad , \tag{46}$$

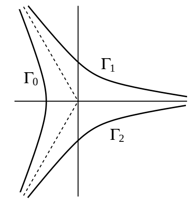


Fig. 1. Contours in the complex u-plane for the Airy functions

where K is a constant which can depend on ξ. In case the moduli of the endpoints of the contour Γ are taken to infinity the standard path-integral representation is recovered. In fact in this case the substitution

$$u = (3\xi)^{-1/3}\psi - \frac{1}{6\xi}$$

leads to

$$R(y;\xi) = K\exp\left(-\frac{1}{108\xi} - \frac{y}{6\xi}\right)\int_\Gamma du\, \exp\left(-\frac{1}{3}u^3 + tu\right).$$

This integral can now be expressed in terms of the Airy functions

$$\int_{\Gamma_j} du\, \exp\left(-\frac{1}{3}u^3 + tu\right) = 2\pi\omega^j\,\mathrm{Ai}(t\omega^j) \quad ,$$

where $\omega = e^{i2\pi/3}$, $j = 0,1,2$ and the contours Γ_j are depicted in Fig. 1. Note that

$$\sum_j \omega^j\,\mathrm{Ai}(t\omega^j) = 0 \quad .$$

For a pure φ^4-theory, similar considerations allow us to express the functions R_j defined in (31), $j = 1,2,3$ as follows

$$R_1 = \frac{1}{\sqrt{\pi}}(2\xi)^{1/4}\exp\left(-\frac{1}{32\xi}\right)\int_{-\infty}^{\infty} d\psi$$
$$\times\exp\left(-\frac{1}{2}\psi^2 - \xi\psi^4 + y\psi\right)$$
$$R_2 = \frac{-i}{\pi^{3/2}}(2\xi)^{1/4}\exp\left(-\frac{1}{32\xi}\right)\int_{-i\infty}^{i\infty} d\psi$$
$$\times\exp\left(-\frac{1}{2}\psi^2 - \xi\psi^4 + y\psi\right) \tag{47}$$

whereas for R_3

$$\left(\int_0^\infty + \int_0^{-i\infty}\right) d\psi\, \exp\left(-\frac{1}{2}\psi^2 - \xi\psi^4 + y\psi\right)$$
$$= \frac{\sqrt{\pi}}{2}(2\xi)^{-1/4}\exp\left(\frac{1}{32\xi}\right)(R_1 - i\pi R_2)$$
$$+\frac{i\pi}{2}\exp\left(\frac{1}{16\xi}\right)(2\xi)^{-1/4}R_3 \quad . \tag{48}$$

Let us have a closer look at the various possible contours in the case of general $\varphi^3+\varphi^4$-theory. Let us denote the various objects in the action as complex numbers:

$$\varphi = |\varphi|e^{i\omega} \quad , \quad \lambda_3 = |\lambda_3|e^{i\eta_3} \quad , \quad \lambda_4 = |\lambda_4|e^{i\eta_4} \quad .$$

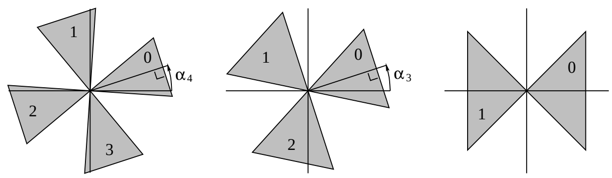


Fig. 2. The regions in the complex φ-plane which correspond with $\Omega_4(k)$, $\Omega_3(k)$ and $\Omega_2(k)$ with $\eta_j = -j\alpha_j$

For simplicity and without loss of generality, we may keep μ real and positive. The direction in the φ plane where the term $\lambda_4\varphi^4$ goes to positive infinity as $|\varphi| \to \infty$ are given by

$$\omega \in \Omega_4(k)\ ,\quad \Omega_4(k) = \left(k\frac{\pi}{2} - \frac{\pi}{8} - \frac{\eta_4}{4}\ ,\ k\frac{\pi}{2} + \frac{\pi}{8} - \frac{\eta_4}{4}\right)\ ,$$
$$k = 0,1,2,3\ .$$

Similarly 'allowed' directions for the $\lambda_3\varphi^3$ term are

$$\omega \in \Omega_3(k)\quad ,$$
$$\Omega_3(k) = \left(k\frac{2\pi}{3} - \frac{\pi}{6} - \frac{\eta_3}{3}\ ,\ k\frac{2\pi}{3} + \frac{\pi}{6} - \frac{\eta_3}{3}\right)\ ,$$
$$k = 0,1,2\ .$$

Finally, the $\mu\varphi^2$ term goes to positive infinity for

$$\omega \in \Omega_2(k)\quad ,\quad \Omega_2(k) = \left(k\pi - \frac{\pi}{4}\ ,\ k\pi + \frac{\pi}{4}\right)\quad , k = 0,1\ .$$

By inspection of these endpoints, already statements can be made about the (non)perturbative character of the theory corresponding to a given contour. To illustrate this, let us consider a pure φ^4-theory, *i.e.* with $\lambda_3 = 0$. Let the contour start at some φ_1, chosen at infinity with argument ω_1, and end at some φ_2, also at infinity in some direction with argument ω_2. These values each have to be in some interval Ω_4: let ω_1 be in $\Omega_4(n_1)$, and ω_2 in $\Omega(n_2)$. We can sufficiently specify the contour by giving n_1 and n_2 so that for instance the contour Γ_{20} for $\eta_4 = 0$ denotes the standard φ^4-theory, where we may take the real line for Γ, start at $\varphi = -\infty$ and end at $\varphi = +\infty$ (interchange of the endpoints corresponds to replacing R by $-R$ and hence does not influence $\phi(x)$). In total, there are six contours that give a viable φ^4-theory: Γ_{01}, Γ_{12}, Γ_{23}, Γ_{30}, Γ_{02} and Γ_{13}. Note that these are related to each other by phase shifts: in fact,

$$\Gamma_{30} = \Gamma_{12}(\eta_4 \to \eta_4 + 2\pi)\ , \Gamma_{23} = \Gamma_{12}(\eta_4 \to \eta_4 + 4\pi)\ ,$$
$$\Gamma_{12} = \Gamma_{12}(\eta_4 \to \eta_4 + 6\pi)\ , \Gamma_{13} = \Gamma_{02}(\eta_4 \to \eta_4 + 2\pi)\ .$$

Therefore, only Γ_{02} and Γ_{01}, say, give really different theories, all other cases being obtainable by an appropriate shift in η_4. All contours, as stated, corresponds to viable theories as long as λ_4 is non-vanishing, but when we let $|\lambda_4| \to 0$ there are two possibilities. It may happen that $\Omega_4(n_1)$ overlaps with one of the Ω_2 segments, and $\Omega_4(n_2)$ with the *other* Ω_2 segment. In that case, the limiting theory is equal to the free theory, and the limit $|\lambda_4| \to 0$ is smooth: we may call this the perturbative limit. In the other case the limit is not smooth, and the path integral R will diverge as $|\lambda_4| \to 0$: we call this the non-perturbative limit. Clearly, the limiting behavior depends on the argument η_4: for the contour Γ_{02} (the 'standard one') one has

$$-\frac{3}{2}\pi < \eta_4 < \frac{3}{2}\pi\ ,\ \frac{5}{2}\pi < \eta_4 < \frac{11}{2}\pi : \text{perturbative}\ ,$$
$$\frac{3}{2}\pi < \eta_4 < \frac{5}{2}\pi\ ,\ \frac{11}{2}\pi < \eta_4 < \frac{13}{2}\pi : \text{non-perturbative}\ ,$$

and for the other contour Γ_{01}:

$$\frac{1}{2}\pi < \eta_4 < \frac{3}{2}\pi\ ,\ \frac{9}{2}\pi < \eta_4 < \frac{11}{2}\pi : \text{perturbative}\ ,$$
$$\frac{3}{2}\pi < \eta_4 < \frac{9}{2}\pi\ ,\ \frac{11}{2}\pi < \eta_4 < \frac{17}{2}\pi : \text{non-perturbative}\ ,$$

For the pure φ^3-theory, there are of course three contours, related to each other: $\Gamma_{20} = \Gamma_{01}(\eta_3 \to \eta_3 + 2\pi)$, $\Gamma_{12} = \Gamma_{01}(\eta_3 \to \eta_3 + 4\pi)$. For the limiting theory we find, for contour Γ_{01}:

$$-\frac{5}{4}\pi < \eta_3 < \frac{1}{4}\pi\ ,\ \frac{7}{4}\pi < \eta_3 < \frac{13}{4}\pi : \text{perturbative}\ ,$$
$$\frac{1}{4}\pi < \eta_3 < \frac{7}{4}\pi\ ,\ \frac{13}{4}\pi < \eta_3 < \frac{19}{4}\pi : \text{non-perturbative}\ .$$

In a theory with both φ^3 and φ^4 couplings, things become more interesting. Of course, as long as λ_4 is nonzero, we are allowed to let λ_3 go to zero without jeopardizing the perturbativity. On the other hand, we can only let λ_4 vanish with fixed λ_3 if the selected Ω_4 and Ω_3 intervals overlap. We give in Fig. 3 and Fig. 4 the values of η_4 and η_3 that correspond to a perturbative $\lambda_4 \to 0$ limit, Finally, we may study the combined limits $\lambda_4 \to 0$ followed by $\lambda_3 \to 0$. The regions of perturbativity are given in Fig. 2. Clearly, these are more restricted since we are in this case requiring a common overlap of Ω_2, Ω_3 and Ω_4.

4 Renormalization

Even though zero-dimensional field theories have no infinities, we may still consider the effects of renormalization, which here take a graph-theoretical significance. Renormalizing the field so that the exact propagator is $1/\mu$, and the coupling constant so that the proper vertices assume their tree-order form, we are counting Green's functions without self-energy and vertex insertions, that is, we are counting the skeleton diagrams of the theory. We shall restrict ourselves to theories that are known to be perturbatively renormalizable in the usual four-dimensional case.

4.1 Renormalization of pure φ^3-theory

In this case renormalization proceeds as usual with the introduction of the tadpole (z_1), the mass (z_2) and the vertex (z_3) counter terms, as well as the corresponding renormalization conditions that imply the dependence of these

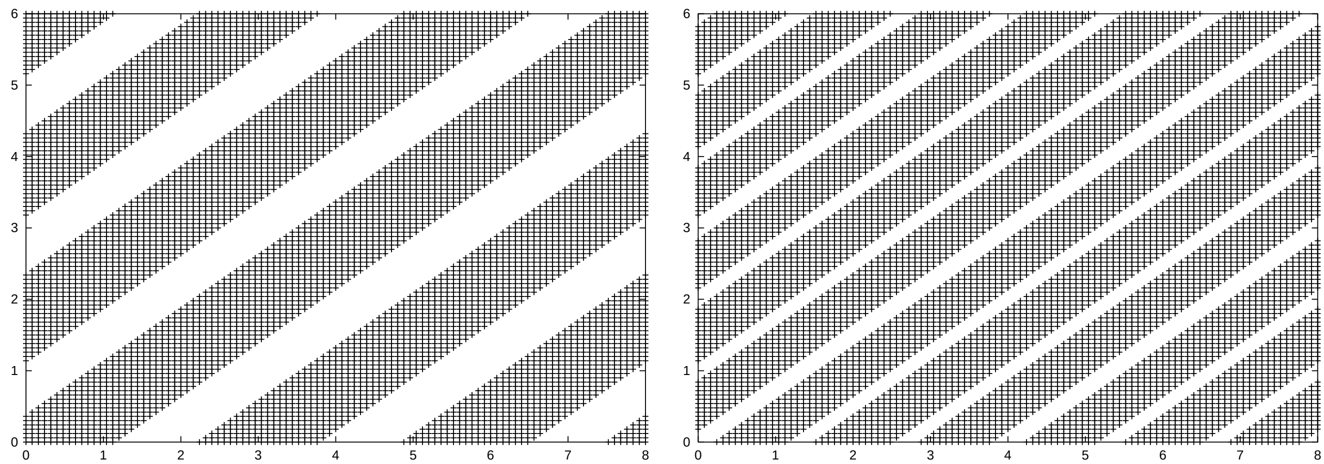


Fig. 3. The shaded areas correspond to combinations of η_4 and η_3 (in units of π) for which the limit $\lambda_4 \to 0$ (with λ_3 fixed) is perturbative, for contour Γ_{01} (left plot) and contour Γ_{02} (right)

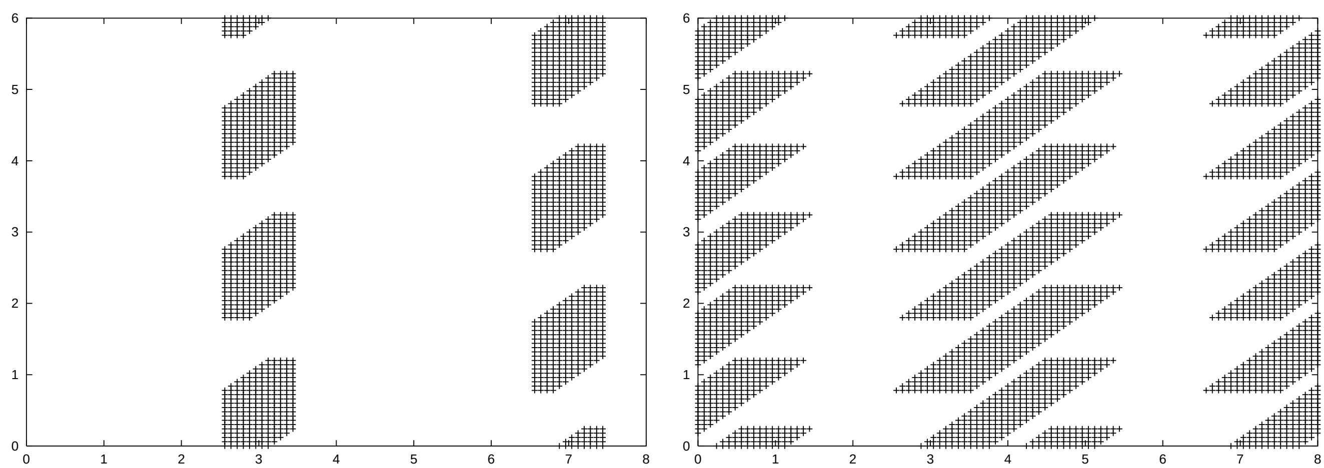


Fig. 4. The shaded areas correspond to combinations of η_4 and η_3 (in units of π) for which the limit $\lambda_4 \to 0$ followed by $\lambda_3 \to 0$ is perturbative, for contour Γ_{01} (left) and Γ_{02}

counter terms on the renormalized coupling constant. The renormalized action can be written as ($\mu = 1, \hbar = 1$)

$$S = \frac{1}{2} z_2 \varphi^2 + \frac{1}{6} g z_3 \varphi^3 + z_1 \varphi \ , \tag{49}$$

and the SD equation takes the form

$$z_2 \phi = (x - z_1) - \frac{G_3}{2}(\phi^2 + \phi') \quad , \qquad G_3 \equiv g z_3 \ . \tag{50}$$

Moreover using (19) we get,

$$3G_3 \frac{\partial \phi}{\partial G_3} = z_2 \phi'' + 2 z_2 \phi \phi' - \phi - (x - z_1)\phi' \ , \tag{51}$$

whereas (6) becomes,

$$\frac{\partial \phi}{\partial z_2} = -\phi \phi' - \frac{1}{2}\phi'' \ . \tag{52}$$

The renormalization conditions that have to be applied are

Condition 4.1.1. No tadpoles, *i.e.* $\phi(x = 0) = 0$;

Condition 4.1.2. propagator $= \phi'(0) = 1$;

Condition 4.1.3. vertex $= \phi''(0) = -g$.

Application of these conditions to the SD equation and its derivative leads to the equations

$$z_1 = -\frac{1}{2} g z_3 \quad , \qquad z_2 = 1 + \frac{1}{2} g^2 z_3 \ . \tag{53}$$

So if we know z_3 as function of g, we know z_1 and z_2 as function of g, and we can consider ϕ to be a function of g and x only. Its derivative w.r.t. g can, using $\partial/\partial z_1 = -\partial/\partial x$, be written as

$$\frac{\partial \phi}{\partial g} = -\phi' \dot{z}_1 + \frac{\partial \phi}{\partial z_2} \dot{z}_2 + \frac{\partial \phi}{\partial G_3} \dot{G}_3 \tag{54}$$

where a dot denotes differentiation w.r.t. g. Because ϕ is a function of x and g only, the l.h.s. is zero in $x = 0$ by

Condition 4.1.1, and evaluation of the r.h.s. leads to the equation

$$\frac{g}{2}\frac{dz_3}{dg} = \frac{2z_3 - 2(1+g^2)z_3^2}{-4+(4+g^2)z_3} \ . \tag{55}$$

It is straightforward to derive that for $g = 0$ the perturbative counter terms read

$$z_1(0) = 0\ ,\quad z_2(0) = 1\ ,\quad z_3(0) = 1\ .$$

(55) is an Abel equation of the second kind [7]. The perturbative solution, satisfying the above initial condition, is

$$z_3(g) = 1 - g^2 - \frac{1}{2}g^4 - 4g^6 - 29g^8 - \frac{545}{2}g^{10} + \cdots\ , \tag{56}$$

an expansion previously given by Cvitanović et al. [4].

Having, however, solved the SD equation for φ^3-theory (Sect. 3.2), we also have an exact, albeit implicit, solution of (55), making use of Condition 4.1.2:

$$\begin{aligned}&[c_1\mathrm{Ai}'(t_0) + c_2\mathrm{Bi}'(t_0)]\left(\frac{2}{gz_3}\right)^{1/3}\\ &\quad -\frac{z_2}{gz_3}[c_1\mathrm{Ai}(t_0) + c_2\mathrm{Bi}(t_0)] = 0\ ,\end{aligned} \tag{57}$$

where

$$\begin{aligned}t_0 &= \left(\frac{2}{gz_3}\right)^{1/3}\left(\frac{1}{2}gz_3 + \frac{z_2^2}{2gz_3}\right)\\ &= \frac{z_2^2}{(2g^2z_3^2)^{2/3}}\left(1 + \frac{g^2z_3^2}{z_2^2}\right)\ ,\end{aligned}$$

and Ai and Bi are the two independent solutions of the Airy equation $f''(t) = tf(t)$. The meaning of this equation (57) is that for a given g and by using (53) as well as the functional form of Ai and Bi we can determine z_3. To show that (57) is an implicit solution of (55), let

$$F(g) = \frac{(2g^2z_3^2)^{1/3}}{z_2}\ ,$$

implying $t_0F(g)^2 - 1 = (gz_3/z_2)^2$, and differentiate (57) with respect to g to get

$$\begin{aligned}&\left(F'(g) - \frac{dt_0}{dg}\right)[c_1\mathrm{Ai}'(t_0) + c_2\mathrm{Bi}'(t_0)]\\ &\quad + F(g)[c_1\mathrm{Ai}''(t_0) + c_2\mathrm{Bi}''(t_0)]\frac{dt_0}{dg} = 0\ .\end{aligned}$$

Using the Airy equation and (57), we get

$$F'(g) + \frac{dt_0}{dg}(t_0F(g)^2 - 1) = 0\ .$$

Explicitly, this says

$$2z_2\frac{d(gz_3)}{dg} - 3gz_3\frac{dz_2}{dg} - 2z_3\frac{d(gz_3)}{dg} = 0\ .$$

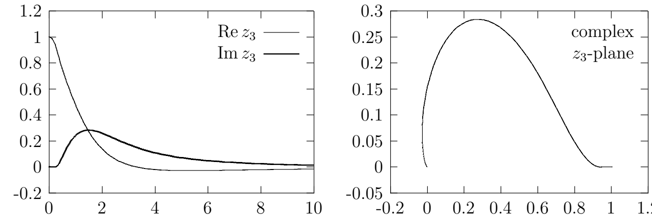


Fig. 5. z_3 as function of $\mathrm{Re}\, g$ with $\mathrm{Im}\, g = 0$ for pure φ^3-theory

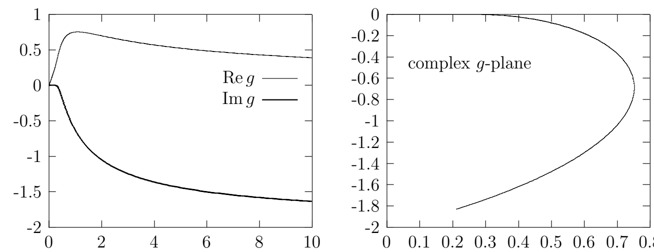


Fig. 6. g as function of $\mathrm{Re}\, gz_3$ with $\mathrm{Im}\, gz_3 = 0$ for pure φ^3-theory

By using (53), one easily sees that the above equation is an equivalent form of (55).

In Fig. 5 we present the results of a numerical calculation of z_3 for the Γ_{10}-contour as function of g, as described in the Appendix. The left graph shows the real and imaginary part of z_3 as function of real and positive values of g. Notice that $z_3(0) = 1$ as demanded, and that the imaginary part does not stay zero for real g. This is, of course, an artifact of the definition of the path integral over a complex contour, which is the Γ_{10}-contour for φ^3-theory in this case (Fig. 2). The right graph combines the real and imaginary part in one curve in the complex z_3-plane.

Fig. 6 shows what happens if we let g run with real and positive values of gz_3, so that the actual coupling constant is real and positive.

4.2 Renormalization of pure φ^4-theory

In the case of φ^4-theory, the renormalized action is given by ($\mu = \hbar = 1$)

$$S = \frac{1}{2}z_2\varphi^2 + \frac{1}{4!}gz_4\varphi^4\ . \tag{58}$$

The SD equation becomes

$$z_2\phi = x - \frac{G_4}{6}(\phi^3 + 3\phi\phi' + \phi'')\ ,\qquad G_4 = gz_4\ , \tag{59}$$

and the stepping equation (19), leads to

$$4G_4\frac{\partial\phi}{\partial G_4} = z_2\phi'' + 2z_2\phi\phi' - \phi - x\phi'\ , \tag{60}$$

whereas (6) assumes the form of (52). The renormalization conditions require that

Condition 4.2.1. $\phi(x=0)=0$;

Condition 4.2.2. $\phi'(0)=1$;

Condition 4.2.3. $\phi'''(0)=-g$,

and application to the SD equation leads to the relation

$$z_2 = 1 - \frac{1}{6}(3-g)gz_4 \quad . \tag{61}$$

As in the case of φ^3-theory, ϕ can be considered to be a function of x and g only, and its derivative w.r.t. g can be written as

$$\frac{\partial\phi}{\partial g} = \frac{\partial\phi}{\partial z_2}\dot{z}_2 + \frac{\partial\phi}{\partial G_4}\dot{G}_4 \quad . \tag{62}$$

The l.h.s. is zero in $x=0$ by Condition 4.2.1, and evaluation of the r.h.s. leads to

$$\frac{dz_4}{dg} = \frac{-6z_4 + (6-9g+3g^2)z_4^2}{6g - g(6-5g+g^2)z_4} \quad , \tag{63}$$

another Abel equation of the second kind. The perturbative solution is given by

$$z_4(g) = 1 + \frac{3}{2}g + \frac{3}{4}g^2 + \frac{11}{8}g^3 - \frac{45}{16}g^4 + \frac{499}{32}g^5 + \cdots \quad . \tag{64}$$

We want to remark at this point that the statement by Cvitanović et al. that, in the case of φ^3-theory, the coefficients of the series expansion of $g - gz_3$ count connected three-point diagrams with no self-energy or vertex insertions cannot be carried foreward to φ^4-theory: the coefficients of the series expansion of $g - gz_4$ do not count connected four-point diagrams with no self-energy or (four-point) vertex insertions. There are, for example, no such diagrams with three vertices.

To find the exact implicit solution of (63), we apply Condition 4.2.2 to the solution (31) of φ^4-theory, resulting in

$$\frac{c_1 t\mathrm{U}(1;t)}{z_2} - \frac{c_2 t\mathrm{V}(1;t)}{z_2\Gamma(\frac{3}{2})} = c_1\mathrm{U}(0;t) + \frac{c_2\mathrm{V}(0;t)}{\Gamma(\frac{1}{2})} \quad ,$$

$$t = \left(\frac{3z_2^2}{gz_4}\right)^{1/2} \quad . \tag{65}$$

Letting

$$F_1(t) = c_1\mathrm{U}(1;t) - \frac{c_2\mathrm{V}(1;t)}{\Gamma(\frac{3}{2})} \quad \text{and}$$

$$F_0(t) = c_1\mathrm{U}(0;t) + \frac{c_2\mathrm{V}(0;t)}{\Gamma(\frac{1}{2})} \, ,$$

the above equation becomes

$$tF_1(t) = z_2 F_0(t) \quad .$$

Using the properties of the parabolic functions we can easily show that

$$F_1'(t) = \left(\frac{z_2}{2} - 1\right)F_0(t) \quad ,$$

$$F_0'(t) = -\frac{1}{2}\left(t + \frac{z_2}{t}\right)F_0(t) \quad ,$$

so that differentiation of (4.2.3) leads to

$$\frac{dt}{dg}\frac{z_2}{t}F_0(t) + t\frac{dt}{dg}\left(\frac{z_2}{2} - 1\right)F_0(t)$$
$$= \frac{dz_2}{dg}F_0(t) - \frac{z_2}{2}\frac{dt}{dg}\left(t + \frac{z_2}{t}\right)F_0(t) \quad .$$

This equation can be written as

$$\frac{d}{dg}\frac{t}{z_2} + \frac{dt}{dg}\left(\frac{1}{2} + \frac{t^2}{z_2^2}\cdot\frac{g-3}{6}gz_4\right) = 0 \quad ,$$

where we used relation (61). Finally, since $t/z_2 = \sqrt{3/gz_4}$ by definition of t, it is easily seen that the above equation becomes (63).

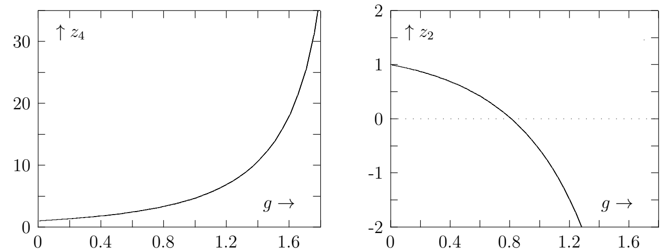


Fig. 7. z_4 as function of g (left) and z_2 as function of g (right)

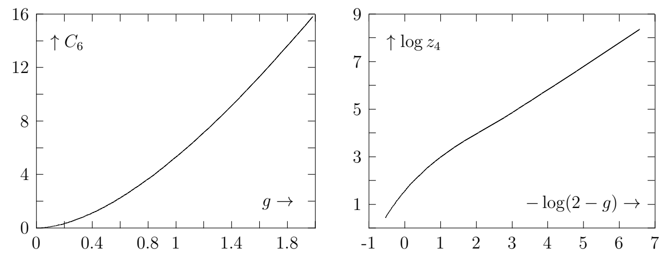


Fig. 8. C_6 as function of g (left) and $\log z_4$ as function of $-\log(2-g)$ (right)

In Fig. 7 we show the results of the numerical calculation of $z_4(g)$, as described in the Appendix. We used the Γ_{20}-contour for φ^4-theory (Fig. 2). Starting at $z_4(0)=1$, $z_4(g)$ stays real and positive for real and positive values of g, as expected. Moreover z_2 exhibits a zero, whose position $g_\star$ can be calculated analytically and is given by

$$g_\star = 3 - \frac{1}{4}\left(\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\right)^2 \sim 0.81155 \quad . \tag{66}$$

At this point the theory becomes 'massless', in the sense that the bare mass becomes zero, yet the Green's functions do not exhibit singular behavior. In fact let us consider the 6-point function as an example. It can be explicitly calculated and it reads

$$C_6 = 6z_4^{-1} - 6 + 9g + g^2 \quad .$$

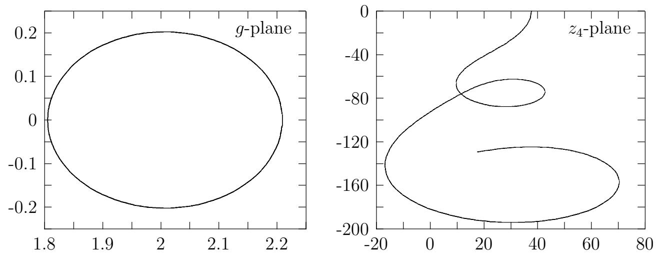


Fig. 9. The complex g-plane and the complex z_4-plane. g goes around twice, anti-clockwise and starting on the real axis on the left side of $g = 2$

It is easy to see that the expansion around $g = 0$ reproduces the known perturbative series. Moreover, the left graph Fig. 8 presents C_6 as a function of g.

We also see that z_4 increases with increasing g, and explodes if g approaches 2. The graph on the right of Fig. 8 suggests that there is a simple pole at $g = 2$. In fact, substitution of a Laurent series around $g = 2$ in (63) results in a solution with a simple pole:

$$z_4(g) = -\frac{6}{g-2} + 6 - 12(g-2) + 54(g-2)^2 \\ -399(g-2)^3 + 3948(g-2)^4 - \cdots \; . \qquad (67)$$

One can ask the question whether this series expansion corresponds to a solution with $z_4(0) = 1$, that is, the perturbative solution. In order to get the perturbative solution from the implicit solution (65), in combination with (61), we should take the constants c_1 and c_2 such that the limit of $t \to \infty$ exists. Using the properties of the parabolic functions and their asymptotic expansions, we find that the perturbative solution has to satisfy

$$z_2 \;=\; t^2 \left(\frac{B_{3/4}(\frac{1}{4}t^2)}{B_{1/4}(\frac{1}{4}t^2)} - 1 \right) \;, \qquad (68)$$

$$B_\nu(\tfrac{1}{4}t^2) := \begin{cases} \frac{1}{\cos\nu\pi}[I_{-\nu}(\frac{1}{4}t^2) + I_\nu(\frac{1}{4}t^2)] & \text{if } \mathrm{Re}\, t < 0 \\ \frac{\pi}{2\sin\nu\pi}[I_{-\nu}(\frac{1}{4}t^2) - I_\nu(\frac{1}{4}t^2)] & \text{if } \mathrm{Re}\, t > 0, \end{cases}$$

together with (61). For g close to, but smaller than, $g = 2$ we see that $z_2 < 0$, so that $\mathrm{Re}\, t < 0$, and it is easy to see that the solution in this case has a simple pole at $g = 2$. However, the coefficients for large powers in the series expansion seem to behave as $(2n+2)!$, so that the series has radius of convergence equal to zero, and the numerical solution of a curve around $g = 2$ in the complex g-plane reveals that there is a branch point (Fig. 9). In any case, for $g \to 2^-$ the bare coupling becomes strong and the bare mass squared large and negative whereas the connected Green's functions are still perfectly calculable; for instance $C_6(g = 2) = 16$.

4.3 Renormalization of $\varphi^3 + \varphi^4$-theory

The renormalization of the φ^3+φ^4-theory is more involved, but straightforward. The action is given by

$$S = \frac{1}{2}z_2\varphi^2 + \frac{1}{3!}G_3\varphi^3 + \frac{1}{4!}G_4\varphi^4 + z_1\varphi \quad ,$$
$$G_3 = g_3 z_3 \quad , \quad G_4 = g_4 z_4 \quad ,$$

and the SD equation assumes the form

$$z_2\phi = (x - z_1) - \frac{G_3}{2}(\phi^2 + \phi') - \frac{G_4}{6}(\phi^3 + 3\phi\phi' + \phi'') \; . \qquad (69)$$

The stepping equations read

$$\frac{\partial\phi}{\partial G_3} = -\frac{1}{6}\phi''' - \frac{1}{2}\phi\phi'' - \frac{1}{2}\phi'^2 - \frac{1}{2}\phi^2\phi'$$
$$\frac{\partial\phi}{\partial G_4} = -\frac{1}{24}\phi'''' - \frac{1}{6}\phi\phi'' - \frac{5}{12}\phi'\phi'' - \frac{1}{4}\phi^2\phi'' \\ - \frac{1}{2}\phi\phi'^2 - \frac{1}{6}\phi^3\phi'$$
$$\frac{\partial\phi}{\partial z_2} = -\phi\phi' - \frac{1}{2}\phi'' \quad ,$$

and the renormalization conditions are now

Condition 4.3.1. $\phi(x = 0) = 0$;

Condition 4.3.2. $\phi'(0) = 1$;

Condition 4.3.3. $\phi''(0) = -g_3$;

Condition 4.3.4. $\phi'''(0) = 3g_3^2 - g_4$.

Combining these conditions with the SD equation one easily gets

$$z_1 = \frac{1}{2}g_3 G_4 - \frac{1}{2}G_3 \quad ,$$
$$z_2 = 1 - \frac{1}{6}(3g_3^2 - g_4 + 3)G_4 + \frac{1}{2}g_3 G_3 \quad , \qquad (70)$$

so that ϕ becomes a function of g_3, g_4 and x only, leading to the the four equations:

$$\left.\frac{\partial\phi}{\partial g_i}\right|_{x=0} \equiv -\phi'(0)\frac{\partial z_1}{\partial g_i} + \left.\frac{\partial\phi}{\partial z_2}\right|_{x=0}\frac{\partial z_2}{\partial g_i} \\ + \left.\frac{\partial\phi}{\partial G_3}\right|_{x=0}\frac{\partial G_3}{\partial g_i} + \left.\frac{\partial\phi}{\partial G_4}\right|_{x=0}\frac{\partial G_4}{\partial g_i} = 0$$

and

$$\left.\frac{\partial\phi'}{\partial g_i}\right|_{x=0} \equiv -\phi''(0)\frac{\partial z_1}{\partial g_i} + \left.\frac{\partial\phi'}{\partial z_2}\right|_{x=0}\frac{\partial z_2}{\partial g_i} \\ + \left.\frac{\partial\phi'}{\partial G_3}\right|_{x=0}\frac{\partial G_3}{\partial g_i} + \left.\frac{\partial\phi'}{\partial G_4}\right|_{x=0}\frac{\partial G_4}{\partial g_i} = 0$$

with $i = 3, 4$. The coefficients at $x = 0$ can be inferred form the stepping equations. This way we have a system of four equations involving the partial derivatives of the functions $G_3(g_3,g_4)$ and $G_4(g_3,g_4)$ with respect to g_3 and g_4. Notice that the equations are linear with respect to the four partial derivatives but higly non-linear with respect to the functions $G_3(g_3,g_4)$ and $G_4(g_3,g_4)$. They can be solved perturbatively with the result

$$\begin{aligned}G_3 &= g_3 - g_3^3\,\hbar + \frac{3}{2}\,g_3\,g_4\,\hbar + 4\,g_3^5\,\hbar^2 - 6\,g_3^3\,g_4\,\hbar^2\\ &+\frac{3}{4}\,g_3\,g_4^2\,\hbar^2 - 4\,g_3^7\,\hbar^3 - \frac{3}{2}\,g_3^5\,g_4\,\hbar^3 + \frac{19}{4}\,g_3^3\,g_4^2\,\hbar^3\\ &+\frac{11}{8}\,g_3\,g_4^3\,\hbar^3 + 7\,g_3^9\,\hbar^4 - \frac{93}{2}\,g_3^7\,g_4\,\hbar^4 + 81\,g_3^5\,g_4^2\,\hbar^4\\ &-\frac{100}{3}\,g_3^3\,g_4^3\,\hbar^4 - \frac{45}{16}\,g_3\,g_4^4\,\hbar^4 + 47\,g_3^{11}\,\hbar^5\\ &-\frac{807}{2}\,g_3^9\,g_4\,\hbar^5 + 927\,g_3^7\,g_4^2\,\hbar^5 - \frac{2787}{4}\,g_3^5\,g_4^3\,\hbar^5\\ &+\frac{1785}{16}\,g_3^3\,g_4^4\,\hbar^5 + \frac{499}{32}\,g_3\,g_4^5\,\hbar^5\\ G_4 &= g_4 + 3\,g_3^4\,\hbar - 6\,g_3^2\,g_4\,\hbar + \frac{3}{2}\,g_4^2\,\hbar - 6\,g_3^6\,\hbar^2 + 5\,g_3^4\,g_4\,\hbar^2\\ &+\frac{3}{2}\,g_3^2\,g_4^2\,\hbar^2 + \frac{3}{4}\,g_4^3\,\hbar^2 + 9\,g_3^8\,\hbar^3 - 43\,g_3^6\,g_4\,\hbar^3\\ &+\frac{151}{2}\,g_3^4\,g_4^2\,\hbar^3 - 39\,g_3^2\,g_4^3\,\hbar^3 + \frac{11}{8}\,g_4^4\,\hbar^3 + 33\,g_3^{10}\,\hbar^4\\ &-324\,g_3^8\,g_4\,\hbar^4 + 834\,g_3^6\,g_4^2\,\hbar^4 - \frac{1485}{2}\,g_3^4\,g_4^3\,\hbar^4\\ &+\frac{1585}{8}\,g_3^2\,g_4^4\,\hbar^4 - \frac{45}{16}\,g_4^5\,\hbar^4 + \frac{1029}{2}\,g_3^{12}\,\hbar^5\\ &-4610\,g_3^{10}\,g_4\,\hbar^5 + \frac{27525}{2}\,g_3^8\,g_4^2\,\hbar^5 - 17020\,g_3^6\,g_4^3\,\hbar^5\\ &+\frac{68595}{8}\,g_3^4\,g_4^4\,\hbar^5 - \frac{10705}{8}\,g_3^2\,g_4^5\,\hbar^5 + \frac{499}{32}\,g_4^6\,\hbar^5\end{aligned}$$

where the $\hbar$ dependence has been restored for convenience.

In the limit $g_3 \to 0$, also $G_3 \to 0$, and the equations reduce to

$$\frac{\partial G_4}{\partial g_3} = 0 \quad , \quad \frac{\partial G_4}{\partial g_4} = \frac{2(2-g_4)G_4^2}{6g_4-(6-5g_4+g_4^2)G_4} \ ,$$
$$\frac{\partial G_3}{\partial g_3} = \frac{G_4}{g_4} \quad , \quad \frac{\partial G_3}{\partial g_4} = 0 \ .$$

Note that $G_4(0,g_4)$ can be identified as $g_4 z_4(g_4)$ where z_4 is the same function is as in pure φ^4-theory. Another interesting result is that the term linear in g_3 in the expansion of G_3 is given by

$$G_3(g_3,g_4) = g_3 z_4(g_4) + \mathcal{O}(g_3^2) \ .$$

4.4 Renormalization of the charged scalar field

In the case of the charged scalar field we consider the integral representation

$$R(x,\bar{x}) = \sqrt{\frac{\mu}{\hbar}}\int d\varphi d\bar{\varphi}\,\exp\left\{-\frac{1}{\hbar}\left[\mu z_2\varphi\bar{\varphi} + \frac{1}{4}\lambda z_4(\varphi\bar{\varphi})^2 - x\bar{\varphi} - \bar{x}\varphi\right]\right\} \ ,$$

which satisfies the SD equation

$$\zeta R'''(\zeta) + 2R''(\zeta) + \frac{2}{g z_4}[z_2 R'(\zeta) - R(\zeta)] = 0 \quad ,$$
$$\zeta = \frac{x\bar{x}}{\mu\hbar} \ , \quad g = \frac{\lambda\hbar}{\mu^2} \ . \tag{71}$$

In the dimensionless variables

$$u = \frac{x}{\sqrt{\mu\hbar}} \ , \quad \bar{u} = \frac{\bar{x}}{\sqrt{\mu\hbar}} \ , \quad \zeta = u\bar{u} \ , \quad \psi = \sqrt{\frac{\mu}{\hbar}}\,\varphi \ ,$$
$$\bar{\psi} = \sqrt{\frac{\mu}{\hbar}}\,\bar{\varphi} \ ,$$

(71) becomes

$$R(u,\bar{u}) = \int d\psi d\bar{\psi}\,\exp\left(-z_2\psi\bar{\psi} - \frac{1}{4}g z_4(\psi\bar{\psi})^2 + u\bar{\psi} + \bar{u}\psi\right),$$

implying

$$\frac{\partial R}{\partial g} = -\frac{1}{4}\frac{d(g z_4)}{dg}\frac{\partial^4 R}{\partial u^2\partial\bar{u}^2} - \frac{dz_2}{dg}\frac{\partial^2 R}{\partial u\partial\bar{u}}$$

or, in terms of the ζ-variable

$$\begin{aligned}\frac{\partial R}{\partial g} &= -\frac{1}{4}\frac{d(g z_4)}{dg}(\zeta^2 R'''' + 4\zeta R''' + 2R'')\\ &-\frac{dz_2}{dg}(\zeta R'' + R') \ .\end{aligned} \tag{72}$$

Now, the generating function of the connected Green's functions is given by

$$\phi(x,\bar{x}) = \hbar\frac{\partial}{\partial\bar{x}}\ln R(\zeta) = \frac{x}{\mu}\frac{R'(\zeta)}{R(\zeta)} \ ,$$

and the renormalization conditions are

Condition 4.4.1. $\frac{\partial\phi}{\partial x}(x=\bar{x}=0) = \frac{1}{\mu}$, implying $R'(0) = R(0)$;

Condition 4.4.2. $\frac{\partial^3\phi}{\partial\bar{x}\partial x^2}(x=\bar{x}=0) = -\frac{\lambda}{\mu^4}$, implying $R''(0) = \left(1-\frac{g}{2}\right)R(0)$.

By combining equations (71) and (72) and the renormalization conditions we get

$$z_2 = 1 - \left(1-\frac{g}{2}\right) g z_4$$

and

$$\frac{dz_4}{dg} = \frac{2z_4 - (3g^2-5g+2)z_4^2}{-2g + g(g^2-3g+2)z_4} \ , \tag{73}$$

with perturbative expansion

$$z_4(g) = 1 + \frac{5}{2}g + \frac{9}{4}g^2 + \frac{49}{8}g^3 - \frac{271}{16}g^4 + \frac{5025}{32}g^5 + \cdots \ . \tag{74}$$

To get an exact implicit solution of (73), we go back to (71) and change variables to get

$$\eta R'''(\eta) + 2R''(\eta) + \alpha^2[R'(\eta) - R(\eta)] = 0 \quad ,$$
$$\eta = \frac{\zeta}{z_2} \quad , \quad \alpha = \sqrt{\frac{2}{gz_4}}\, z_2 \quad . \tag{75}$$

This equation has exactly the form of (43), and the perturbative solution is given by

$$R(\eta) = \sum_{n=0}^{\infty} \frac{(\eta\alpha)^n}{n!}\, \mathrm{U}(n+\tfrac{1}{2};\alpha) \quad .$$

Condition 4.4.1 implies the implicit exact solution of the form

$$\alpha \mathrm{U}(\tfrac{3}{2};\alpha) = z_2 \mathrm{U}(\tfrac{1}{2};\alpha) \quad , \tag{76}$$

where, of course, $z_2 = 1 - (1-g/2)gz_4$.

To show that (76) is indeed an implicit solution of (73), differentiate (76) with respect to g:

$$[\mathrm{U}(\tfrac{3}{2};\alpha) + \alpha \mathrm{U}'(\tfrac{3}{2};\alpha)]\frac{d\alpha}{dg} = \frac{dz_2}{dg}\mathrm{U}(\tfrac{1}{2};\alpha) + z_2\mathrm{U}'(\tfrac{1}{2};\alpha)\frac{d\alpha}{dg} \quad ,$$

and using parabolic cylinder functions properties together with (76) to get

$$\frac{dz_2}{dg} = \left(\frac{z_2}{\alpha} + \alpha z_2 + \frac{z_2^2}{\alpha} - \alpha\right)\frac{d\alpha}{dg} \quad , \tag{77}$$

with $\alpha = \sqrt{\frac{2}{gz_4}}\, z_2$. It is a straight forward calculation to show that this is indeed (73).

In the following we present a derivation of the initial condition for $z_4(g=0)$. Using the path integral expression of (71), we find the SD equation

$$\mu z_2 \hbar \bar\partial R + \frac{\lambda z_4}{2}\hbar^3 \partial \bar\partial^2 R - xR = 0 \quad .$$

The generating function $\phi = \hbar\bar\partial \ln R$ of the connected diagrams satisfies

$$\phi(0,0) = 0 \quad , \quad (\bar\partial\phi)(0,0) = 0 \quad ,$$
$$\bar\phi(0,0) = 0 \quad , \quad (\partial\bar\phi)(0,0) = 0 \quad .$$

as well as the SD equation

$$\phi = \frac{x}{\mu z_2} - \frac{\lambda z_4}{2\mu z_2}(\bar\phi\phi^2 + 2\hbar\phi\partial\phi + \bar\phi\bar\partial\phi + \hbar^2\bar\partial\partial\phi) \quad ,$$

with the renormalization conditions 4.4.1 and 4.4.2. These should hold for any value of $\hbar$, and for $\hbar = 0$, the SD equation becomes

$$\phi_0 = \frac{x}{\mu z_2(0)} - \frac{\lambda z_4(0)}{2\mu z_2(0)}\bar\phi_0\phi_0^2 \quad ,$$

from which we derive for the perturbative solution that

$$(\partial\phi_0)(0,0) = \frac{1}{\mu z_2(0)} \qquad \Rightarrow \qquad z_2(0) = 1 \quad ,$$
$$(\bar\partial\partial^2\phi_0)(0,0) = -\frac{\lambda z_4(0)}{\mu^4} \qquad \Rightarrow \qquad z_4(0) = 1 \quad .$$

Notice that the value $\hbar = 0$ is directly related to $g = 0$ since g is proportional to $\hbar$.

5 Summary

In this paper we studied several aspects of zero-dimensional field theories. In the first place we derived a set of diagrammatic equations, including the well known Schwinger-Dyson equations as well as a set of 'stepping' equations generalizing some previous results. Then we showed how to solve these equations exactly in terms of known functions and we established integral representations of these solutions, best known as the 'path integral' representation. Explicit results were obtained for φ^3, φ^4, $\varphi^3+\varphi^4$ and the charged scalar field theories. Subsequently, we studied the 'renormalization' of such theories in zero dimensions, which is equivalent to counting diagrams with restrictions imposed on the type of diagrams considered, for instance diagrams without any tadpoles, self-energy insertions or vertex insertions. We were able to get explicit results for the dependence of the bare quantities such as the mass, the coupling, and the tadpole counter terms, on the renormalized (physical) coupling constant. Examples of interesting observations are the facts that in φ^4 theory, the bare mass exhibits a zero at a finite value of the renormalized coupling constant $g = g_\star$ ((66)), whereas at $g \to 2-\epsilon$ the bare coupling becomes strong and the mass squared becomes large and negative. Yet in both cases the 'physical' connected Green's functions remain finite and calculable.

Appendix

Consider general φ^p-theory, and suppose that all but one renormalization conditions have been implemented through functions $z_k(g,z_p)$, $k = 1,2,..,p-1$ of two variables z_p and g, like in (53) and (61). This means that we are considering a theory with an action

$$S(g,z_p;\varphi) = \frac{gz_p}{p!}\,\varphi^p + \sum_{k=1}^{p-1}\frac{z_k(g,z_p)}{k!}\,\varphi^k \quad .$$

Let Γ be a contour in the complex φ-plane, such that $\mathrm{Re}\,\varphi^p \to \infty$ at the endpoints, and define

$$Z_n(g,z_p) := \int_\Gamma d\varphi\,\varphi^n\,\exp\{-S(g,z_p;\varphi)\} \quad .$$

Such an integral is not defined for all complex values of gz_p. Let $gz_p = |gz_p|e^{ip\eta}$ and denote by $e^{-i\eta}\Gamma$ the contour that is obtained from Γ by clockwise rotation over η. For complex values of gz_p, we define

$$\begin{aligned} Z_n(g,z_p) :=& \int_{e^{-i\eta}\Gamma} d\varphi\,\varphi^n\,\exp\{-S(g,z_p;\varphi)\} \\ =&\ e^{-i(n+1)\eta}\int_\Gamma d\varphi\,\varphi^n \\ &\times\exp\left(-\frac{|gz_p|}{p!}\,\varphi^p - \sum_{k=1}^{p-1}\frac{z_k}{k!}\,e^{-ik\eta}\varphi^k\right) \quad . \end{aligned}$$

Integrals of this type can easily be calculated to high precision by numerical integration. One just has to choose Γ such that it goes through one or more saddle points, so that the integrand oscillates as little as possible.

To formulate the renormalization problem further, let us denote the connected moments by C_n, so

$$C_1 = \frac{Z_1}{Z_0}\ , \quad C_2 = \frac{Z_2}{Z_0} - C_1^2\ ,$$
$$C_3 = \frac{Z_3}{Z_0} - 3C_2C_1 - C_1^3\ , \ldots$$

and so on. The problem is to solve z_p as function of g from the implicit function equation

$$C_p(g, z_p) = -g \ \ ,$$

which represents the final renormalization condition. This equation can be solved numerically. Given a value of g, we have to find the zero of the function

$$F(z_p) := C_p(g, z_p) + g \ \ ,$$

which can be found using Newton-Raphson iteration

$$z_p \leftarrow z_p - \frac{F(z_p)}{F'(z_p)} \ \ .$$

By making small steps in the value of g, the solution $z_p(g)$ on a curve in the complex g-plane can be determined. At the start of each iteration, the question arises of which initial value of z_p to choose, and the obvious answer is to choose the final value of the previous iteration, which should lie close the the new final value if the steps in g are not to large.

As a check one can look whether the results obtained with this method satisfy (numerically) the available differential equations for $z_p(g)$ ((55) and (63)).

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